Denavit-Hartenberg notation for common robots

Peter Corke

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1 Introduction

Denavit-Hartenberg parameters are one of the most confusing topics for those new to the study of robotic arms. This note discusses some common robot configurations and the physical meaning of their various Denavit-Hartenberg parameters. Consistent diagrams and tables of Denavit-Hartenberg parameters are used to illustrate the main points.

Fundamentally we wish to describe the pose of each link in the chain relative to the pose of the preceding link. We would expect this to comprise **six parameters**, one of which is the joint variable — the parameter of the joint that connects the two links. However the Denavit-Hartenberg formalism[1, Ch. 7] uses only **four parameters** to describe the spatial relationship between successive link coordinate frames, and this is achieved by introducing two constraints[2, p. 78] to the placement of those frames:

1. The axis x_j is perpendicular to the axis z_{j-1}.
2. The axis x_j intersects the axis z_{j-1}.

The result is that link frames are sometimes constrained to be placed in locations that seem non-obvious, perhaps not even on the physical link itself. The choices of coordinates frames are also not unique, different people will derive different, but correct, coordinate frame assignments[2]. These variants will however always lead to the same expression for the pose of the end-effector with respect to the base.

In robot kinematics it is common to partition the joints into two sets

$$\boldsymbol{T}_6 = \boldsymbol{T}_p(q_1 \cdots q_3)\boldsymbol{T}_o(q_4 \cdots q_6) \tag{1.1}$$

The first transform, a function of the first three joints, controls the position of the origin of the coordinate frame $\{P\}$ and its responsible for setting the position of the frame $\{6\}$.

The second transform, a function of the last three joints, controls the orientation of the frame $\{6\}$ with respect to frame $\{P\}$. This transform is a pure rotation, and on modern robots is implemented by a **spherical wrist** mechanism that provides an arbitrary orientation with zero translation — the origin of the frames $\{P\}$ and $\{6\}$ are coincident. The wrist will be discussed in more detail in Section 5.

This approach allows us to decouple position and orientation of the end-effector $\{6\}$. The orientation of frame $\{P\}$ is a function of the first three joints, but we use those degrees of freedom to control position, not orientation. Thus the orientation of $\{P\}$ is a consequence of the position we are trying to achieve. The job of the wrist is to provide a rotation from $\{P\}$ to $\{6\}$ as required by the task.

In practice things are not quite so clear cut. The robot's tool, perhaps the fingertips of its gripper, must be accounted for. The robot may not be positioned at the origin of the world coordinate frame, and could be hanging upside down from the ceiling. In some cases the $\boldsymbol{T}_o$ term contains a finite translation, violating the clean decoupling that is often assumed. This short note attempts to explain some of the complexities using a few well known robot types for illustration.

Notation

- The coincident origin of the frames $\{P\}$ and $\{6\}$ are indicated by a large black dot in the figures that follow. This is the centre of the wrist and the position of the end-effector.
- The x-, y- and z-axes of the base and $\{P\}$ are denoted by the colored arrows that are red, blue and green respectively. Joint rotation axes are denoted by dashed lines.
- For tables of Denavit-Hartenberg parameters rows are colored blue if they contribute to wrist position, and orange if they contribute to wrist orientation. White cells are related to base or tool transforms.

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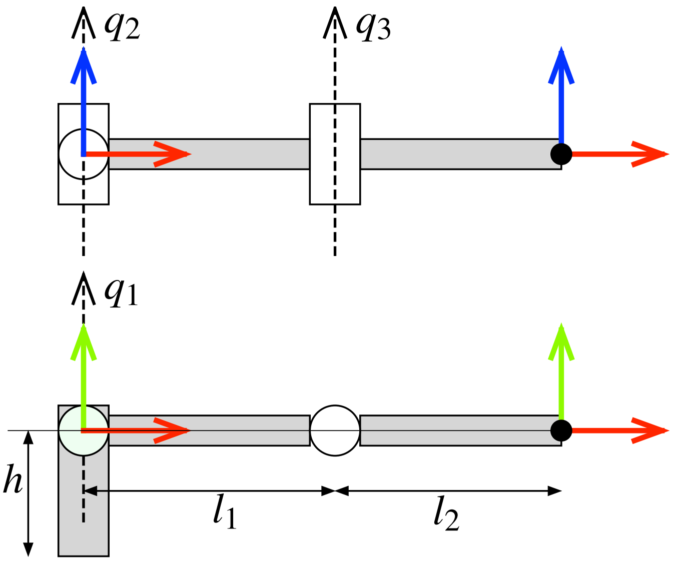


(a) Plan (top) and side elevation (bottom)

j	$\theta_\mathbf{j}$	$\mathbf{d_j}$	$\mathbf{a_j}$	$\alpha_\mathbf{j}$
1	q_1	0	0	$-\pi/2$
2	q_2	0	l_1	0
3	q_3	0	l_2	$\pi/2$
4	q_4	0	0	$-\pi/2$
5	q_5	0	0	$\pi/2$
6	q_6	0	0	0

(b) Denavit-Hartenberg parameters

Figure 1: Simple anthropomorphic robot with no shoulder offset.

Almost all robot arms are anthropomorphic, human like, and they have:

1. a waist rotation around a vertical axis,
2. a shoulder rotation about a horizontal axis,
3. an elbow rotation about a horizontal axis,
4. and a 3-axis wrist joint at the end.

A plan and side elevation of such a robot is shown in Figure 1a with the joint angles $q_1 = q_2 = q_3 = 0$. The wrist joints are not shown, but the location of the wrist is shown by the solid black dot. The coordinate frame directions of the base and the endpoint are shown as colored arrows. The notation RRR refers to the three joints being all revolute.

The Denavit-Hartenberg parameters of this robot are given in Figure 1b where the blue color is associated with the first three joints (wrist position), and the orange color is associated with the last three joints (the wrist). The joint variables $q_1 \cdots q_6$ appear in the first column, the θ column, since this robot has all revolute joints.

This robot has only two significant dimensions, the lengths of the upper and lower arm segments which are denoted l_1 and l_2 respectively. They appear in the a column. All other a_j and d_j values are zero.

The pose of $\{P\}$ is given by

$$\boldsymbol{T}_p = \boldsymbol{R}_z(q_1)\boldsymbol{R}_x(-\frac{\pi}{2})\boldsymbol{R}_z(q_2)\boldsymbol{T}_x(l_1)\boldsymbol{R}_z(q_3)\boldsymbol{T}_x(l_2)\boldsymbol{R}_x(\frac{\pi}{2})$$

From inspection of Figure 1a we could directly write[3] a simpler sequence of transforms

$$\boldsymbol{T}_p = \boldsymbol{R}_z(q_1)y\boldsymbol{R}_y(q_2)\boldsymbol{T}_x(l_1)\boldsymbol{R}_y(q_3)\boldsymbol{T}_x(l_2)$$

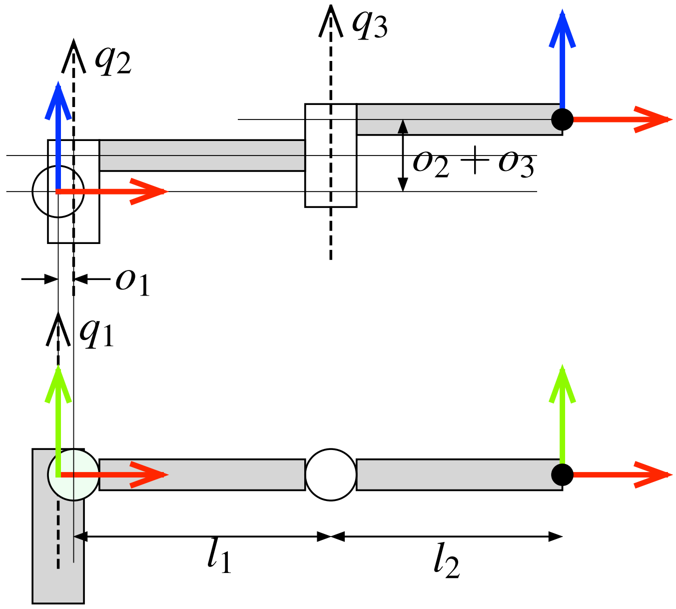
which uses only 5, rather than 7, elementary transforms — the extra complexity in the first sequence is due to the constraints introduced by the Denavit-Hartenberg formalism.

Note the values of α_1 and α_3. They always have the same magnitude, $\pi/2$, but opposite signs. There is no clear convention and different kinematic models treat this differently. In this note we always consider that $\alpha_1 = -\pi/2$ unless otherwise noted. The consequence for changing the order are:

- the direction of the rotational axes for joints 2 and 3, the dashed arrows are reversed.
- the expression for pose (2) will be different, and the applicable inverse kinematic solution will be different.

3 Arms with a shoulder offset

3.1 RRR arm general case



(a) Plan (top) and side elevation (bottom)

j	θ_j	d_j	a_j	α_j
1	q_1	0	o_1	$-\pi/2$
2	q_2	o_2	l_1	0
3	q_3	o_3	l_2	$\pi/2$
4	q_4	0	0	$-\pi/2$
5	q_5	0	0	$\pi/2$
6	q_6	0	0	0

(b) Denavit-Hartenberg parameters

Figure 2: Simple anthropomorphic robot with a shoulder offset.

Many robots have the arm assembly horizontally to one side of the base x-axis, as shown in Figure 2a. Examples include the PUMA 560 and most of the Fanuc and ABB arms. Such robots have a left and right-handed working configuration[1, Sec 7.4.4] — the configuration shown in Figure 2a is left handed. However by changing joints 1 and 2 by 180° the arm will be in a right-handed configuration.

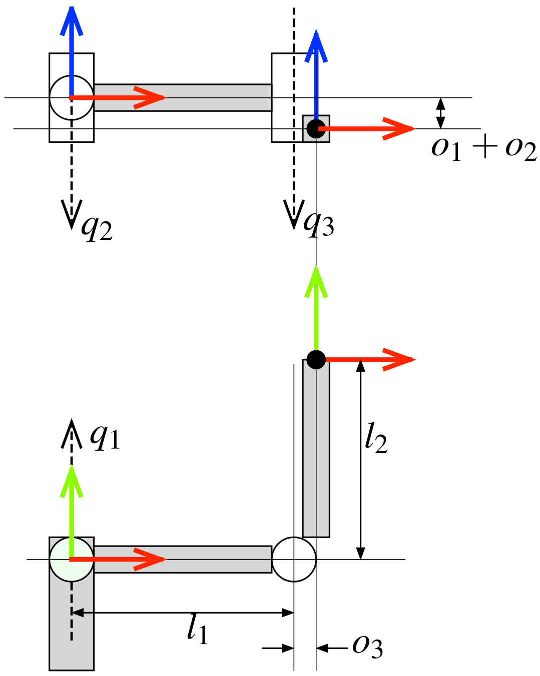
This robot has two significant dimensions, the lengths of the upper and lower arm segments which are denoted l_1 and l_2 respectively as before. It also has three offsets denoted o_1, o_2 and o_2 respectively. Either or both of o_2 and o_2 will cause the end frame to be displaced in the $\{P\}$ y-direction. One introduces the shift at the shoulder, the other at the elbow. The choice of how the offset is partitioned depends on where you choose to physically place the coordinate frames — it makes no difference to the end-point position.

Offset o_1 is less common and is typically zero. As shown in the diagram it implies that the rotational axes of joints 1 and 2 do not intersect.

The pose of $\{P\}$ is given by

$$\boldsymbol{T}_p = \boldsymbol{R}_z(q_1)\boldsymbol{T}_x(o_1)\boldsymbol{R}_x(-\frac{\pi}{2})\boldsymbol{R}_z(q_2)\boldsymbol{T}_z(o_2)\boldsymbol{T}_x(l_1)\boldsymbol{R}_z(q_3)\boldsymbol{T}_z(o_3)\boldsymbol{T}_x(l_2)\boldsymbol{R}_x(\frac{\pi}{2})$$

3.2 RRR special case: PUMA 560 robot



(a) Plan (top) and side elevation (bottom)

j	θ_j	$\mathbf{d_j}$	$\mathbf{a_j}$	α_j
1	q_1	0	0	$+\pi/2$
2	q_2	o_1	l_1	0
3	q_3	o_2	o_3	$-\pi/2$
4	q_4	l_2	0	$\pi/2$
5	q_5	0	0	$-\pi/2$
6	q_6	0	0	0

(b) Denavit-Hartenberg parameters[4]

Figure 3: PUMA 560 anthropomorphic robot with a shoulder offset.

The classical PUMA 560 robot is a special case of the above, but there are some important differences:

- α_1 and α_3 have opposite signs to the other models discussed, and this changes the direction of the rotational axes for joints 2 and 3 — the dashed arrows are reversed.
- the robot has a very clear shoulder offset and in the Denavit-Harteberg parameters it can be expressed using the o_1 and o_2 offsets described above. However in most Puma kinematic models $o_1 = 0$ and the offset is lumped into the elbow joint rather than the shoulder.
- there is an oddity in the mechanical design, an elbow offset — the centre lines of the upper and lower arms do not intersect. In the side elevation this is shown in exagerated fashion by the offset o_3. This offset is described by a_3 which we used previously to express the lower arm length l_2.

  At this point we've run out of parameters in the first three joints, so we need to pull a rather unintuitive trick. We consider the upper arm pointing vertically for the case where $q_3 = 0$, and we use one of the joint 4 (wrist) parameters d_4 to represent the lower arm length l_2.

> The parameters of the PUMA 560 robot are shown in Figure 3b and the color coding now clearly shows that the wrist position and orientation is not clearly split between the first and last three joints.

The pose of $\{P\}$ is more complex in this case, we must take into account the first **four** joints

$$ {}^0\boldsymbol{T}_4 = \boldsymbol{R}_z(q_1)\boldsymbol{R}_x(\frac{\pi}{2})\boldsymbol{R}_z(q_2)\boldsymbol{T}_z(o_1)\boldsymbol{T}_x(l_1)\boldsymbol{R}_z(q_3)\boldsymbol{T}_z(o_2)\boldsymbol{T}_x(o_3)\boldsymbol{R}_x(-\frac{\pi}{2})\boldsymbol{R}_z(q_4)\underbrace{\boldsymbol{T}_z(l_2)}\boldsymbol{R}_x(-\frac{\pi}{2}) $$

The indicated term for lower-arm length term can be moved one place to the left since translation along z is invariant to rotation about z

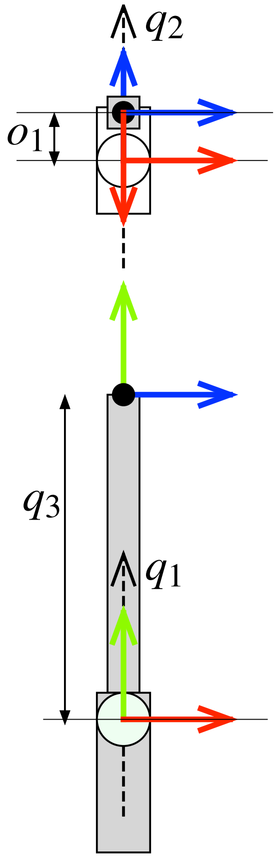
$$ {}^0\boldsymbol{T}_4 = \boldsymbol{R}_z(q_1)\boldsymbol{R}_x(\frac{\pi}{2})\boldsymbol{R}_z(q_2)\boldsymbol{T}_z(o_1)\boldsymbol{T}_x(l_1)\boldsymbol{R}_z(q_3)\boldsymbol{T}_z(o_2)\boldsymbol{T}_x(o_3)\boldsymbol{R}_x(-\frac{\pi}{2})\boldsymbol{T}_z(l_2)\underbrace{\boldsymbol{R}_z(q_4)\boldsymbol{R}_x(-\frac{\pi}{2})} $$

The last two terms affect only orientation, not position, allowing us to write a transform expression for the position of $\{P\}$

$$ \boldsymbol{T}_p = \boldsymbol{R}_z(q_1)\boldsymbol{R}_x(\frac{\pi}{2})\boldsymbol{R}_z(q_2)\boldsymbol{T}_z(o_1)\boldsymbol{T}_x(l_1)\boldsymbol{R}_z(q_3)\boldsymbol{T}_z(o_2)\boldsymbol{T}_x(o_3)\boldsymbol{R}_x(-\frac{\pi}{2})\boldsymbol{T}_z(l_2) $$

Note that the indicated terms must be taken into account when considering the overall orientation of the robot.

3.3 RRP arm with prismatic joint



(a) Plan (top) and side elevation (bottom)

j	θ_j	$\mathbf{d_j}$	$\mathbf{a_j}$	α_j
1	q_1	0	0	$-\pi/2$
2	q_2	o_1	0	$+\pi/2$
3	$-\pi/2$	q_3	0	0
4	q_4	0	0	$-\pi/2$
5	q_5	0	0	$+\pi/2$
6	q_6	0	0	0

(b) Denavit-Hartenberg parameters[5, **?**]

Figure 4: Stanford robot with a shoulder offset and prismatic joint.

The Stanford arm is a not really an anthropomorphic arm. It has a waist and a shoulder like the other robots that we have discussed, but the length of the arm is variable — it has a sliding, or prismatic, third joint. There is also a shoulder offset which is handled by the offset parameter o_1.

The Denavit-Hartenberg formalism requires that the prismatic extension be in the z-direction, it is the d_3 parameter so in this case $d_3 = q_3$. Therefore we are required to draw the arm in a vertical configuration.

Note that there is a somewhat arbitrary rotation, $\theta_3 = -\pi/2$, which makes the y-axis of $\{P\}$ point to the right — in all the other robot models we consider it is the x-axis that points to the right. This rotation is described in [**?**] but is zero in [5].

The pose of $\{P\}$ is given by

$$\boldsymbol{T}_p = \boldsymbol{R}_z(q_1)\boldsymbol{R}_x(-\frac{\pi}{2})\boldsymbol{R}_z(q_2)\boldsymbol{T}_z(o_1)\boldsymbol{R}_z(-\frac{\pi}{2})\boldsymbol{T}_z(q_3)$$

4 Base and tool transforms

4.1 Base transform

The origin of the robot's coordinate system is mechanically at the intersection of joints 1 and 2 axes[1], a point typically inside the shoulder mechanism. The value of d_1 is typically set to zero, but most real robots are mounted on some kind of pedestal above the floor. This parameter could be used to set the height of the q_2 axis at some arbitrary height h above the ground plane, as shown in Figure 1a.

If the robot is at some arbitrary pose with respect to a world coordinate system then we can rewrite (1.1) as

$$\boldsymbol{T}_6 = \boldsymbol{T}_B\boldsymbol{T}_p(q_1 \cdots q_3)\boldsymbol{T}_o(q_4 \cdots q_6)$$

where $\boldsymbol{T}_B$ is the robot's base transform.

4.2 Tool transform

The origin of the robot's $\{6\}$ coordinate frame is at the intersection of the joints 4, 5 and 6 axes, a point typically in the centre of the wrist assembly — physically inside the mechanism. A useful robot carries a tool, the end of which carries the frame $\{E\}$ which is at some fixed pose with respect to the frame $\{6\}$. The tip of the tool, the origin of $\{E\}$, is often referred to as the Tool Centre Point (TCP).

We can rewrite (1.1) as

$$\boldsymbol{T}_E = \boldsymbol{T}_B\boldsymbol{T}_p(q_1 \cdots q_3)\boldsymbol{T}_o(q_4 \cdots q_6)\boldsymbol{T}_T$$

where $\boldsymbol{T}_B$ is the robot's base transform already mentioned, and $\boldsymbol{T}_T$ is the tool transform that describes the relative pose of the end-effector frame $\{E\}$ with respect to the wrist centre frame $\{6\}$.

5 The wrist

Recapping from the introduction, it is common in robot kinematics to partition the joints into two sets as given by (1.1)

$$\boldsymbol{T}_6 = \boldsymbol{T}_p(q_1 \cdots q_3)\boldsymbol{T}_o(q_4 \cdots q_6)$$

[1] Note that for some robots these two axes do not intersect

The first transform, a function of the first three joints, controls the pose of the coordinate frame $\{P\}$ and its responsible for setting the position of the end-effector frame $\{E\}$. The second transform, a function of the last three joints, controls the orientation of the end-effector frame $\{E\}$ with respect to frame $\{P\}$. This transform is a pure rotation, and on modern robots is implemented by a **spherical wrist** mechanism that provides an arbitrary orientation with zero translation — the origin of the frames $\{P\}$ and $\{E\}$ are coincident This approach allows us to decouple position and orientation of the end-effector $\{E\}$.

In practice we wish to generalise the kinematic expression by introducing a base and tool transform

$$\boldsymbol{T}_E = \boldsymbol{T}_B \boldsymbol{T}_p(q_1 \cdots q_3) \boldsymbol{T}_o(q_4 \cdots q_6) \boldsymbol{T}_T$$

so that the tip of the robot's tool is now the coordinate frame $\{E\}$ which is some useful distance away from the origin of frame $\{6\}$ which is physically inside the root's wrist mechanism.

The ideal spherical wrist, rotation only, is described by the Denavit-Hartenberg parameters shown in orange

j	$\theta_\mathbf{j}$	$\mathbf{d_j}$	$\mathbf{a_j}$	$\alpha_\mathbf{j}$
4	q_4	0	0	$-\pi/2$
5	q_5	0	0	$\pi/2$
6	q_6	0	0	0

The key parameters are:

- α_4 and α_5 have magnitudes of $\pi/2$ but opposite signs. For some kinematic models they are $-\pi/2, +\pi/2$ respectively, in others they are $+\pi/2, -\pi/2$ respectively.
- All the length parameters d_j and a_j are zero, to ensure pure rotation.

However the more general case for joints 4, 5 and 6 is

j	$\theta_\mathbf{j}$	$\mathbf{d_j}$	$\mathbf{a_j}$	$\alpha_\mathbf{j}$
4	q_4	d_4	0	$-\pi/2$
5	q_5	0	0	$\pi/2$
6	q_6	z_t	x_t	α_t

which contains several non-zero translations.

Consider first the case where d_4 is non-zero, and this arises in the kinematic model of the PUMA 560. Because the z-axis of the $\{3\}$ frame points into the wrist mechanism, we can rearrange the transform string so that the translation due to d_4 comes before the q_4 rotation (see Section 3.2). In this case, and assuming $a_4 = 0$, we can write

$$\boldsymbol{A}_4 = \boldsymbol{R}_z(q_4) \boldsymbol{T}_z(d_4) \boldsymbol{R}_x(\alpha_4)$$

and rearrange it as

$$\boldsymbol{A}_4 = \underbrace{\boldsymbol{T}_z(d_4)}_{\boldsymbol{T}_{d4}} \underbrace{\boldsymbol{R}_z(q_4)\boldsymbol{R}_x(\alpha_4)}_{\boldsymbol{A}'_4}$$

recognising that the translation d_4 but it can be transposed with the proceeding rotational term in the Denavit-Hartenberg factor and treated as a standalone translation.

Consider now the case where any or all of d_6, a_6 and α_6 are non zero. There are three parameters in the last row of the Denavit-Hartenberg parameter table that we have previously shown in white. We can write

$$\boldsymbol{A}_6 = \underbrace{\boldsymbol{R}_z(q_6)}_{\boldsymbol{A}'_6} \underbrace{\boldsymbol{T}(x_t, 0, z_t)\boldsymbol{R}_x(\alpha_t)}_{\boldsymbol{T}_t}$$

as the final wrist rotation, followed by a transform from the wrist centre which represents a limited tool transform that may be sufficient for some cases. Note that for the case when $d_6 = a_6 = \alpha_6 = 0$ then

$$\boldsymbol{A}_6 = \boldsymbol{R}_z(q_6) = \boldsymbol{A}'_6$$

We can expand (1.1) as

$$\boldsymbol{T}_E = \boldsymbol{T}_B \boldsymbol{A}_1 \boldsymbol{A}_2 \boldsymbol{A}_3 \boldsymbol{A}_4 \boldsymbol{A}_5 \boldsymbol{A}_6 \boldsymbol{T}_T$$

and then substitute the above equations for $\boldsymbol{A}_4$ and $\boldsymbol{A}_6$ to write

$$\boldsymbol{T}_E = \boldsymbol{T}_B \underbrace{\boldsymbol{A}_1 \boldsymbol{A}_2 \boldsymbol{A}_3 \boldsymbol{T}_{d4}}_{\boldsymbol{T}_p} \underbrace{\boldsymbol{A}'_4 \boldsymbol{A}_5 \boldsymbol{A}'_6}_{\boldsymbol{R}_w} \boldsymbol{T}_t \boldsymbol{T}_T \tag{5.1}$$

which we can group into a position, orientation and tool transforms. The link transforms $\boldsymbol{A}'_4$ and $\boldsymbol{A}'_6$ are pure rotations.

The robot tool transform is now overspecified as

$$\boldsymbol{T}_t \boldsymbol{T}_T = \boldsymbol{T}(x_t, 0, z_t)\boldsymbol{R}_x(\alpha_t)\boldsymbol{T}_T$$

and the user can choose whether to embed the tool transform into the link 6 parameters or keep the wrist parameters 'pure' and use a separate general transform $\boldsymbol{T}_T$.

Now we solve (5.1) for the required pure wrist rotation

$$\boldsymbol{R}_w = \boldsymbol{T}_p^{-1} \boldsymbol{T}_B^{-1} \boldsymbol{T}_E \boldsymbol{T}_T^{-1} \boldsymbol{T}_t^{-1}$$

5.1 Solving for spherical wrist angles

Given that we have solved for the required value of the pure wrist rotation $\boldsymbol{R}_w$ we can now determine the three wrist joint angles: q_4, q_5 and q_6. The wrist kinematic model contains two link twist angles α_4 and α_5. These both have a magnitude of $\pi/2$ but opposite signs so there are two ways to describe the spherical wrist.

For the case $\alpha_4 = -\pi/2$ and $\alpha_5 = +\pi/2$ we can write a string of elementary transformations

$$\boldsymbol{R}_w = \boldsymbol{R}_z(q_4)\, \underbrace{\boldsymbol{R}_x(-\frac{\pi}{2})\boldsymbol{R}_z(q_5)\boldsymbol{R}_x(\frac{\pi}{2})}\, \boldsymbol{R}_z(q_6)$$

which is a sequence of pure rotations, so the overall transform is a pure rotation — there is no translation. The indicated term can be simplified as

$$\boldsymbol{R}_x(-\frac{\pi}{2})\boldsymbol{R}_z(q_5)\boldsymbol{R}_x(\frac{\pi}{2}) = \boldsymbol{R}_y(q_5)$$

which gives the wrist rotation as

$$\boldsymbol{R}_w = \boldsymbol{R}_z(q_4)\boldsymbol{R}_y(q_5)\boldsymbol{R}_z(q_6)$$

which is the ZYZ Euler angle sequence. The wrist mechanism is a physical instantiation of the Euler angle sequence.

Given $\boldsymbol{R}_w$ we can solve for this using the Robotics Toolbox function `tr2eul()`. For many robots the first wrist joint q_4 has a large operating range, close to 360 degrees, and that allows for two solutions of q_4 separated by 180 degrees. These are often referred to as the "no flip" and "flip" configuration respectively. The flip solution can be obtained by passing the option `'flip'` to `tr2eul()`.

For the case $\alpha_4 = +\pi/2$ and $\alpha_5 = -\pi/2$ we can write a string of elementary transformations

$$\boldsymbol{R}_w = \boldsymbol{R}_z(q_4)\, \underbrace{\boldsymbol{R}_x(\frac{\pi}{2})\boldsymbol{R}_z(q_5)\boldsymbol{R}_x(-\frac{\pi}{2})}_{\boldsymbol{R}_y(q_5)}\, \boldsymbol{R}_z(q_6) \quad (5.2)$$

$$= \boldsymbol{R}_z(q_4)\boldsymbol{R}_y(-q_5)\boldsymbol{R}_z(q_6) \quad (5.3)$$

which is also a ZYZ Euler angle sequence but with the middle angle negated.

6 Conclusions

Inverse kinematics has many complexities because the Denavit-Hartenberg parameters for any robot can be written in several different ways.

The treatment of inverse kinematics in contempory texts[5][2][6][7] typically focuses on a few simple examples. The general case of offsets is covered, if at all, in a cursory manner. The effect of certain Denavit-Hartenberg parameters such as base height d_1, mid arm translation d_4 and the tool transform d_6, a_6 and α_6 are rarely covered. Nor is the effect of making changing the first or fourth link twist α_1 from $-\pi/2$ to $\pi/2$, which actually requires a new solution to be derived from scratch.

References

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