Solving trigonometric equations

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When solving problems in inverse kinematics we often end up with an equation of the form. There are several ways to solve this and they are discussed below.

Simple derivation

$$a \cos\theta + b \sin\theta = c, \;\; a, b \neq 0 \tag{1}$$

and we note similarity with the form of the sum of angles identity

$$\underbrace{\sin\phi}_{\sim a} \cos\theta + \underbrace{\cos\phi}_{\sim b} \sin\theta \equiv \sin(\phi + \theta)$$

In order to equate coefficients we need to ensure that $\sin^2\phi + \cos^2\phi = 1$ which is true only if $a^2 + b^2 = 1$. In general this will not be the case so we normalize the equation, dividing each side by $d = \sqrt{a^2 + b^2}$ giving

$$a' \cos\theta + b' \sin\theta = c' \tag{2}$$

where $a' = a/d$, $b' = b/d$ and $c' = c/d$. Now we can write

$$\sin\phi = a', \, \cos\phi = b'$$

and solve for ϕ

$$\tan\phi = \frac{a'}{b'} = \frac{a}{b} \in [-2\pi, 2\pi)$$

which should be computed using an `atan2` function.
Next we rewrite (2) as

$$\sin(\phi + \theta) = c'$$

and if $|c'| \leq 1$ or $a^2 + b^2 - c^2 > 0$ we can solve for

$$\theta = \sin^{-1} c' - \phi$$

In general, there is a second solution corresponding to the negative solution of the square root $d = -\sqrt{a^2+b^2}$ leading to

$$\tan\phi = \frac{-a'}{-b'} = \frac{-a}{-b} \in [-2\pi, 2\pi)$$

which puts the solution for ϕ in the diagonally opposite quadrant, and

$$\sin(\phi+\theta) = -c'$$

Since $\sin(-x) = -\sin(x)$ we can write the second solution as

$$\theta = -\sin^{-1} c' - \phi$$

In summary, the two solutions are

$$\begin{aligned} \theta &= \sin^{-1} c' - \phi, \quad \tan\phi = \tfrac{a}{b} \\ \theta &= -\sin^{-1} c' - \phi, \quad \tan\phi = \tfrac{-a}{-b} \end{aligned}$$

We can test this numerically using MATLAB

```
a = 4; b = 5; c = 3;
clear theta

phi = atan2(a, b);
th1 = asin(c/norm([a b])) - phi
```

```
th1 = -0.1871
```

```
phi = atan2(-a, -b);
th2 = -asin(c/norm([a b])) - phi;
theta = [th1 th2];
a*cos(theta) + b*sin(theta) - c
```

```
ans = 1x2
1.0e+-15 *
    -0.8882    -0.8882
```

which indicates solutions equal to zero up to machine precision.

Other forms

Another commonly given solutions of this equation include

$$\theta = \tan^{-1}\frac{c}{\pm\sqrt{a^2+b^2-c^2}} - \tan^{-1}\frac{a}{b}$$

and

$$\theta = \tan^{-1}\frac{bc \pm ad}{ac \mp bd}$$

which requires just a single arc-tangent operation. See the discussion at https://math.stackexchange.com/questions/213545/solving-trigonometric-equations-of-

The Weierstrass transformation

A well known way to convert trigonometric equations to algebraic equations is with the Weierstrass transformation[1] which is familiar as one of the half-angle identities

$$\sin\theta = \frac{2h}{1+h^2}, \quad \cos\theta = \frac{1-h^2}{1+h^2}$$

where $h = \tan\frac{\theta}{2}$. The problem (1) can be rewritten as

$$\frac{2bh}{h^2+1} - \frac{a(h^2-1)}{h^2+1} = c$$

which can be expressed as a quadratic in h

$$(a+c)h^2 - 2bh + a + c = 0$$

which we can solve as

$$h = \begin{pmatrix} \frac{b+\sqrt{a^2+b^2-c^2}}{a+c} \\ \frac{b-\sqrt{a^2+b^2-c^2}}{a+c} \end{pmatrix} \tag{3}$$

and clearly has the two solutions so long as $a^2 + b^2 - c^2 > 0$. The transformation has led to the solution in a very straightforward fashion.
Once again, we can test this numerically using MATLAB

```
syms a b c h theta
e = a*cos(theta) + b*sin(theta) == c
```

e =

$$a\cos(\theta) + b\sin(\theta) = c$$

```
e = rewrite(e, 'tan')
```

e =

$$\frac{2b\tan\left(\frac{\theta}{2}\right)}{\tan\left(\frac{\theta}{2}\right)^2+1} - \frac{a\left(\tan\left(\frac{\theta}{2}\right)^2-1\right)}{\tan\left(\frac{\theta}{2}\right)^2+1} = c$$

```
e = subs(e, tan(theta/2), h)
```

e =

$$\frac{2bh}{h^2+1} - \frac{a\left(h^2-1\right)}{h^2+1} = c$$

[1] After the German mathematician Karl Weierstrass (1815-1897).

```
sol = solve(e, h)
```

sol =

$$\begin{pmatrix} \frac{b+\sqrt{a^2+b^2-c^2}}{a+c} \\ \frac{b-\sqrt{a^2+b^2-c^2}}{a+c} \end{pmatrix}$$

Now lets validate this

```
a = 4; b = 5; c = 3;
theta = 2*atan(eval(sol))
```

```
theta = 2x1
    1.9792
   -0.1871
```

```
a*cos(theta) + b*sin(theta) - c
```

```
ans = 2x1
1.0e+-15 *
         0
   -0.8882
```

which again is equal to zero up to machine precision.