Solving trigonometric equations

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When solving problems in inverse kinematics we often end up with an equation of the form. There are several ways to solve this and they are discussed below.

Simple derivation

\[ a \cos \theta + b \sin \theta = c, \quad a, b \neq 0 \]  

(1)

and we note similarity with the form of the sum of angles identity

\[ \sim_a \sin \phi \cos \theta + \cos \phi \sin \theta \equiv \sin(\phi + \theta) \sim_b \]

In order to equate coefficients we need to ensure that \( \sin^2 \phi + \cos^2 \phi = 1 \) which is true only if \( a^2 + b^2 = 1 \). In general this will not be the case so we normalize the equation, dividing each side by \( d = \sqrt{a^2 + b^2} \) giving

\[ a' \cos \theta + b' \sin \theta = c' \]  

(2)

where \( a' = a/d, \ b' = b/d \) and \( c' = c/d \). Now we can write

\[ \sin \phi = a', \cos \phi = b' \]

and solve for \( \phi \)

\[ \tan \phi = \frac{a'}{b'} = \frac{a}{b} \in [-2\pi, 2\pi] \]

which should be computed using an \texttt{atan2} function.

Next we rewrite (2) as

\[ \sin(\phi + \theta) = c' \]

and if \( |c'| \leq 1 \) or \( a'^2 + b'^2 - c'^2 > 0 \) we can solve for

\[ \theta = \sin^{-1} c' - \phi \]
In general, there is a second solution corresponding to the negative solution of the square root \( d = -\sqrt{a^2 + b^2} \) leading to

\[
\tan \phi = \frac{-a'}{-b'} = \frac{-a}{-b} \in [-2\pi, 2\pi)
\]

which puts the solution for \( \phi \) in the diagonally opposite quadrant, and

\[
\sin(\phi + \theta) = -c'
\]

Since \( \sin(-x) = -\sin(x) \) we can write the second solution as

\[
\theta = -\sin^{-1} c' - \phi
\]

In summary, the two solutions are

\[
\begin{align*}
\theta &= \sin^{-1} c' - \phi, \quad \tan \phi = \frac{a}{b} \\
\theta &= -\sin^{-1} c' - \phi, \quad \tan \phi = \frac{-a}{b}
\end{align*}
\]

We can test this numerically using MATLAB

```matlab
a = 4; b = 5; c = 3;
clear theta
phi = atan2(a, b);
th1 = asin(c/norm([a b])) - phi

th1 = -0.1871

phi = atan2(-a, -b);
theta = [th1 th2];
a*cos(theta) + b*sin(theta) - c

ans = 1x2
1.0e+15 *
-0.8882 - 0.8882
```

which indicates solutions equal to zero up to machine precision.

**Other forms**

Another commonly given solutions of this equation include

\[
\theta = \tan^{-1} \frac{c}{\pm\sqrt{a^2 + b^2 - c^2}} - \tan^{-1} \frac{a}{b}
\]

and

\[
\theta = \tan^{-1} \frac{bc \pm ad}{ac \mp bd}
\]

which requires just a single arc-tangent operation. See the discussion at [https://math.stackexchange.com/questions/213545/solving-trigonometric-equations-of-](https://math.stackexchange.com/questions/213545/solving-trigonometric-equations-of-)
The Weierstrass transformation

A well known way to convert trigonometric equations to algebraic equations is with the Weierstrass transformation\(^1\) which is familiar as one of the half-angle identities

\[
\sin \theta = \frac{2h}{1+h^2}, \quad \cos \theta = \frac{1-h^2}{1+h^2}
\]

where \(h = \tan \frac{\theta}{2}\). The problem (1) can be rewritten as

\[
\frac{2bh}{h^2+1} - \frac{a(h^2-1)}{h^2+1} = c
\]

which can be expressed as a quadratic in \(h\)

\[
(a + c)h^2 - 2bh + a + c = 0
\]

which we can solve as

\[
h = \left( \frac{b + \sqrt{a^2+b^2-c^2}}{a+c} \right) \left( \frac{a+c}{b - \sqrt{a^2+b^2-c^2}} \right)
\]

and clearly has the two solutions so long as \(a^2 + b^2 - c^2 > 0\). The transformation has led to the solution in a very straightforward fashion. Once again, we can test this numerically using MATLAB

\[
\text{syms a b c h theta}
\]

\[
e = \text{a*cos(theta)} + \text{b*sin(theta)} = = \text{c}
\]

\[
e = a \cos (\theta) + b \sin (\theta) = c
\]

\[
e = \text{rewrite(e, ’tan’)}
\]

\[
e = \frac{2b \tan \left( \frac{\theta}{2} \right)}{\tan \left( \frac{\theta}{2} \right)^2 + 1} - \frac{a \left( \tan \left( \frac{\theta}{2} \right)^2 - 1 \right)}{\tan \left( \frac{\theta}{2} \right)^2 + 1} = c
\]

\[
e = \text{subs(e, tan(theta/2), h)}
\]

\[
e = \frac{2bh}{h^2+1} - \frac{a(h^2-1)}{h^2+1} = c
\]

\(^1\)After the German mathematician Karl Weierstrass (1815-1897).
\[ \text{sol} = \text{solve}(e, h) \]

\[ \text{sol} = \begin{pmatrix} \frac{b + \sqrt{a^2 + b^2 - c^2}}{a+c} \\ \frac{b - \sqrt{a^2 + b^2 - c^2}}{a+c} \end{pmatrix} \]

Now lets validate this

\[ a = 4; \ b = 5; \ c = 3; \theta = 2*\text{atan}(	ext{eval(sol)}) \]

\[ \theta = 2 \times 1 \]
\[ 1.9792 \]
\[ -0.1871 \]

\[ a*\cos(\theta) + b*\sin(\theta) - c \]

\[ \text{ans} = 2 \times 1 \]
\[ 1.0 \times 10^{-15} * \]
\[ 0 \]
\[ -0.8882 \]

which again is equal to zero up to machine precision.