SLAM EKF: the insertion Jacobian

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The EKF formulation for mapping and SLAM requires that the state vector is extended every time a new landmark is discovered. For the map-making case the state vector is

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_m \end{pmatrix} \in \mathbb{R}^{n \times 1}$$

where $\boldsymbol{x}_m \in \mathbb{R}^{2m \times 1}$ and m is the number of landmarks so $n = 2m$. For the SLAM case the state vector is

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{x}_v \\ \boldsymbol{x}_m \end{pmatrix} \in \mathbb{R}^{n \times 1}$$

where $\boldsymbol{x}_v \in \mathbb{R}^{3 \times 1}$ is the vehicle configuration so $n = 2m + 3$.

1 Mapping case

The state vector is extended by the function

$$\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{z}) = \begin{matrix} \scriptstyle n \\ \scriptstyle 2 \end{matrix} \underset{1}{\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{g}(\boldsymbol{x}_v, \boldsymbol{z}) \end{pmatrix}}$$

which simply the concatenation the two vectors, and where $\boldsymbol{z}$ is the observation. The Jacobians of this function are

$$\boldsymbol{Y}_x = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \begin{pmatrix} \partial \boldsymbol{x} / \partial \boldsymbol{x} \\ \partial \boldsymbol{g} / \partial \boldsymbol{x} \end{pmatrix} = \begin{matrix} \scriptstyle n \\ \scriptstyle 2 \end{matrix} \underset{n}{\begin{pmatrix} \boldsymbol{1}_{n \times n} \\ \boldsymbol{0}_{2 \times n} \end{pmatrix}} \in \mathbb{R}^{n+2 \times n}$$

$$\boldsymbol{Y}_z = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{z}} = \begin{pmatrix} \partial \boldsymbol{x} / \partial \boldsymbol{z} \\ \partial \boldsymbol{g} / \partial \boldsymbol{z} \end{pmatrix} = \begin{matrix} \scriptstyle n \\ \scriptstyle 2 \end{matrix} \underset{2}{\begin{pmatrix} \boldsymbol{0}_{n \times 2} \\ \boldsymbol{G}_z \end{pmatrix}} \in \mathbb{R}^{n \times 2}$$

The zero element of $\boldsymbol{Y}_x$ is because the state vector comprises only landmark coordinates and $g(\cdot)$ is a function of vehicle configuration which is not part of the state vector.

The linearized covariance update, as discussed in Setion H.2, is

$$\boldsymbol{Y}_x \boldsymbol{P} \boldsymbol{Y}_x^T + \boldsymbol{Y}_z \boldsymbol{W} \boldsymbol{Y}_z^T$$

and substituting the Jacobians we can write

$$= \begin{matrix} \color{red}{n} \\ \color{red}{2} \end{matrix} \underset{\color{red}{n \quad 2}}{\begin{pmatrix} \boldsymbol{P} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \mathbf{0}_{2\times 2} \end{pmatrix}} + \begin{matrix} \color{red}{n} \\ \color{red}{2} \end{matrix} \underset{\color{red}{n \quad 2}}{\begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \boldsymbol{G}_z \boldsymbol{W} \boldsymbol{G}_z^T \end{pmatrix}} \tag{1}$$

$$= \begin{pmatrix} \boldsymbol{P} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \boldsymbol{G}_z \boldsymbol{W} \boldsymbol{G}_z^T \end{pmatrix} \tag{2}$$

which, using the procedure of Appendix A, can be shown to be equivalent to the quadratic expression

$$\begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \boldsymbol{G}_z \end{pmatrix} \begin{pmatrix} \boldsymbol{P} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \boldsymbol{W} \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \boldsymbol{G}_z \end{pmatrix}^T$$

where

$$\begin{matrix} \color{red}{n} \\ \color{red}{2} \end{matrix} \underset{\color{red}{n \quad 2}}{\begin{pmatrix} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times 2} \\ \mathbf{0}_{2\times n} & \boldsymbol{G}_z \end{pmatrix}}$$

is the insertion Jacobian.

2 SLAM case

The state vector is extended by the function

$$\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{z}) = \begin{matrix} \color{red}{n} \\ \color{red}{2} \end{matrix} \underset{\color{red}{1}}{\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{g}(\boldsymbol{x}_v, \boldsymbol{z}) \end{pmatrix}}$$

which simply the concatenation the two vectors, and where $\boldsymbol{z}$ is the observation. Recall that the state vector contains the vehicle configuration

$$\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{z}) = \begin{matrix} \color{red}{3} \\ \color{red}{2m} \\ \color{red}{2} \end{matrix} \underset{\color{red}{1}}{\begin{pmatrix} \boldsymbol{x}_v \\ \boldsymbol{x}_m \\ \boldsymbol{g}(\boldsymbol{x}_v, \boldsymbol{z}) \end{pmatrix}}$$

The Jacobians of this function are

$$\begin{aligned} \boldsymbol{Y}_x = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} &= \begin{pmatrix} \partial \boldsymbol{x}_v / \partial \boldsymbol{x}_v & \partial \boldsymbol{x}_v / \partial \boldsymbol{x}_m \\ \partial \boldsymbol{x}_m / \partial \boldsymbol{x}_v & \partial \boldsymbol{x}_m / \partial \boldsymbol{x}_m \\ \partial \boldsymbol{g} / \partial \boldsymbol{x}_v & \partial \boldsymbol{g} / \partial \boldsymbol{x}_m \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1}_{3\times 3} & \mathbf{0}_{3\times 2m} \\ \mathbf{0}_{2m\times 3} & \mathbf{1}_{2m\times 2m} \\ \boldsymbol{G}_{xv} & \mathbf{0}_{2\times 2m} \end{pmatrix} = \begin{matrix} \color{red}{n} \\ \color{red}{2} \end{matrix} \underset{\color{red}{n}}{\begin{pmatrix} \mathbf{1}_{n\times n} \\ \boldsymbol{G}_x \end{pmatrix}} \in \mathbb{R}^{n+2\times n} \end{aligned}$$

where

$$\boldsymbol{G}_x = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} \in \mathbb{R}^{2\times n}$$

which accounts for the fact that the first three elements of $\boldsymbol{x}$ are the vehicle configuration.

$$
\boldsymbol{Y}_z = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{z}} = \begin{pmatrix} \partial \boldsymbol{x} / \partial \boldsymbol{z} \\ \partial \boldsymbol{g} / \partial \boldsymbol{z} \end{pmatrix} = \begin{array}{c} n \\ 2 \end{array} \underset{2}{\begin{pmatrix} \boldsymbol{0}_{n \times 2} \\ \boldsymbol{G}_z \end{pmatrix}} \in \mathbb{R}^{n \times 2}
$$

The linearized covariance update, as discussed in Setion H.2, is

$$
\boldsymbol{Y}_x \boldsymbol{P} \boldsymbol{Y}_x^T + \boldsymbol{Y}_z \boldsymbol{W} \boldsymbol{Y}_z^T
$$

and substituting the Jacobians we can write

$$
= \begin{array}{c} n \\ 2 \end{array} \underset{n \qquad 2}{\begin{pmatrix} \boldsymbol{P} & \boldsymbol{P}\boldsymbol{G}_x^T \\ \boldsymbol{G}_x \boldsymbol{P} & \boldsymbol{G}_x \boldsymbol{P} \boldsymbol{G}_x^T \end{pmatrix}} + \begin{array}{c} n \\ 2 \end{array} \underset{n \qquad 2}{\begin{pmatrix} \boldsymbol{0}_{n \times n} & \boldsymbol{0}_{n \times 2} \\ \boldsymbol{0}_{2 \times n} & \boldsymbol{G}_z \boldsymbol{W} \boldsymbol{G}_z^T \end{pmatrix}} \tag{3}
$$

$$
= \begin{array}{c} n \\ 2 \end{array} \underset{n \qquad 2}{\begin{pmatrix} \boldsymbol{P} & \boldsymbol{P}\boldsymbol{G}_x^T \\ \boldsymbol{G}_x \boldsymbol{P} & \boldsymbol{G}_x \boldsymbol{P} \boldsymbol{G}_x^T + \boldsymbol{G}_z \boldsymbol{W} \boldsymbol{G}_z^T \end{pmatrix}} \tag{4}
$$

which, using the procedure of Appendix A, can be shown to be equivalent to the quadratic expression

$$
\begin{pmatrix} \boldsymbol{1}_{n \times n} & \boldsymbol{0}_{n \times 2} \\ \boldsymbol{G}_x & \boldsymbol{G}_z \end{pmatrix} \begin{pmatrix} \boldsymbol{P} & \boldsymbol{0}_{n \times 2} \\ \boldsymbol{0}_{2 \times n} & \boldsymbol{W} \end{pmatrix} \begin{pmatrix} \boldsymbol{1}_{n \times n} & \boldsymbol{0}_{n \times 2} \\ \boldsymbol{G}_x & \boldsymbol{G}_z \end{pmatrix}^T
$$

where

$$
\begin{array}{c} n \\ 2 \end{array} \underset{n \qquad 2}{\begin{pmatrix} \boldsymbol{1}_{n \times n} & \boldsymbol{0}_{n \times 2} \\ \boldsymbol{G}_x & \boldsymbol{G}_z \end{pmatrix}}
$$

is the insertion Jacobian.

A Factoring the Jacobian

Consider a quadratic expression

$$
\boldsymbol{M} \begin{pmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W} \end{pmatrix} \boldsymbol{M}^T
$$

where $\boldsymbol{M}$ is a general block-structured matrix

$$
\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}
$$

which we can expand as

$$
\begin{pmatrix} \boldsymbol{A}\boldsymbol{P}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{W}\boldsymbol{B}^T & \boldsymbol{A}\boldsymbol{P}\boldsymbol{C}^T + \boldsymbol{B}\boldsymbol{W}\boldsymbol{D}^T \\ \boldsymbol{C}\boldsymbol{P}\boldsymbol{A}^T + \boldsymbol{D}\boldsymbol{W}\boldsymbol{B}^T & \boldsymbol{C}\boldsymbol{P}\boldsymbol{C}^T + \boldsymbol{D}\boldsymbol{W}\boldsymbol{D}^T \end{pmatrix}
$$

Now, given a matrix of the form

$$
\begin{pmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_z \boldsymbol{W} \boldsymbol{G}_z^T \end{pmatrix}
$$

we can equate coefficients to determine that $\boldsymbol{A} \equiv 1$, $\boldsymbol{B} \equiv 0$, $\boldsymbol{C} \equiv 0$ and $\boldsymbol{D} \equiv \boldsymbol{G}_z$ leading to a factorization as

$$
\begin{pmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_z \end{pmatrix} \begin{pmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W} \end{pmatrix} \begin{pmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_z \end{pmatrix}^T
$$